Explicit form of the gauge transformation generator in terms of constraints

1999 J. Phys. A: Math. Gen. 32 8185

(http://iopscience.iop.org/0305-4470/32/46/314)

J. Phys. A: Math. Gen. **32** (1999) 8185–8195. Printed in the UK PII: S0305-4470(99)02260-X

Explicit form of the gauge transformation generator in terms of constraints

A Shirzad† and M Shabani Moghadam‡

† Department of Physics, Isfahan University of Technology, Isfahan, Iran
‡ Institute for Studies in Theoretical Physics and Mathematics, PO Box 5746, Tehran, 19395, Iran

E-mail: shirzad@cc.iut.ac.ir

Received 3 March 1999, in final form 2 September 1999

Abstract. We study the conditions that a gauge transformation generator should satisfy. Then considering a special class of constrained systems, we have derived the explicit form of the gauge generator in terms of first-class constraints of the system. We have shown that the generator derived can be demonstrated by two different known methods, and satisfies the conditions required in both methods.

1. Introduction

Gauge invariance plays a crucial role in most developments of theoretical physics. Dirac was the first who showed that a gauge theory is indeed a constrained system [1]. Although a large number of gauge-invariant theories have been studied by physicists, there is still not a closed formulation to find the complete set of gauge transformations of a given arbitrary system. Some attempts have been made in this direction, both in Hamiltonian [2–6] and Lagrangian [7, 8] formulations.

In the Hamiltonian formulation, one expects that gauge symmetry, like any other symmetry, should be described by a generator which we call a *gauge generator*. The explicit form of the gauge generator can be given just for very simple constraint systems, but not for the general case. However, there are some methods to get close to it, as far as possible. We will discuss two existing methods in section 2. As we will see, in each method, some conditions need to be imposed on the coefficients of expansion of the gauge generator. Going through the calculations, one observes that, in each method, finding explicit solutions for the coefficients of the gauge generator is *practically* impossible; and at most one can give a proof for the existence of the solution.

In this paper we consider a special class of constrained systems (see relations (17) and (18) below) and solve the equations regarding the coefficients of the gauge generator in one of the methods (section 3). In this way we derive an explicit form of the gauge generator. Thereafter we deform it into the shape of the gauge generator of the other method, and show that it satisfies the requirements of that method. In section 4 we present an example to show what happens during the calculations. Section 5 is devoted to some concluding remarks.

0305-4470/99/468185+11$30.00

2. Gauge generator

2.1. Gauge transformation

Dynamical symmetry transformation (DST) is defined as any transformation which transforms a solution of equations of motion into another solution. In this paper we define a gauge transformation (GT) as a DST which contains arbitrary functions of time. Suppose $(q(t), p(t))$ is a solution of Hamilton–Dirac equations of motion. This means that there exist Lagrange multipliers $\lambda^a(t)$ $a = 1, \ldots, M$ such that the time evolution equation of any function $g(q, p)$ reads [1]

$$\dot{g}(q, p) = [g, H] + \lambda^a(t)[g, \Phi_0^a] \tag{1}$$

where the symbol [,] denotes Poisson brackets, H is the canonical Hamiltonian and Φ_0^a are the set of primary constraints and summation over the repeated indices is understood. Suppose that the gauge-transformed trajectory

$$\begin{aligned}\bar{q}(t) &= q(t) + \delta q(t)\\ \bar{p}(t) &= p(t) + \delta p(t)\end{aligned} \tag{2}$$

is another solution of the equations of motion in phase space. We assume that one can derive the infinitesimal gauge variation of phase space coordinates via the action of the gauge generator $G(q, p, t)$:

$$\begin{aligned}\delta q(t) &= [q, G(q, p, t)]\\ \delta p(t) &= [p, G(q, p, t)].\end{aligned} \tag{3}$$

As stated above, for variation (3) to be a gauge transformation, the generator $G(q, p, t)$ *should* contain some arbitrary functions of time.

On the other hand, the Lagrange multipliers $\lambda^a(t)$ are not phase space coordinates, and their gauge transformation cannot be defined directly from the generator G. Therefore, in order to complete the gauge transformation (2), we assume that the gauge-transformed Lagrange multipliers

$$\bar{\lambda}^a(t) = \lambda^a(t) + \delta\lambda^a(t) \tag{4}$$

can be defined in a suitable way, such that the whole set of variables $(\bar{q}, \bar{p}, \bar{\lambda})$ also satisfy equation (1), as follows:

$$\dot{g}(\bar{q}, \bar{p}) = [g(\bar{q}, \bar{p}), H(\bar{q}, \bar{p})] + \bar{\lambda}^a(t)[g(\bar{q}, \bar{p}), \Phi_0^a(\bar{q}, \bar{p})]. \tag{5}$$

It has been shown [2, 4, 5] that the necessary and sufficient conditions for a function $G(q, p, t)$ to be the generator of a gauge transformation, are as follows:

$$\begin{aligned}&G = \mathrm{FC}\\ &[H, G] - \frac{\partial G}{\partial t} \cong \mathrm{PFC}\\ &[G, \mathrm{PFC}] \cong \mathrm{PFC}\end{aligned} \tag{6}$$

where FC means *first class* and PFC means *primary first-class constraint* and the symbol $\cong$ denotes a Dirac strong equality.

The main problem is to find a function $G(q, p, t)$ with the above properties. Assuming that the Dirac conjecture [1] is valid, one expects that the complete set of first-class constraints of the system should somehow be the building blocks of the generator G. Two distinct methods are proposed in this regard, so far, which will be discussed later in this section. However, due to its close relationship with gauge symmetry, let us first give a few words about the constraint structure.

2.2. Constraint structure

As is well known [1], primary constraints are consequences of the definition of momenta. The primary constraints should be valid over the course of time, so their time derivatives should vanish. This leads to second-level constraints. The same thing should also be considered for second- and higher-level constraints. Suppose the consistency of primary constraints from (1) lead to Φ_1^a, the consistency of Φ_1^a leads to Φ_2^a, and so on.

We assume for convenience that the system is first class. This means that, using consistency conditions, none of the Lagrange multipliers could be determined in terms of phase space coordinates. Hence, using the equation (1) for constraints of any level, the second term on the right-hand side should vanish weakly. This is possible if the Poisson bracket of constraints of any level with the primary constraints vanish on the surface of the constraints known up to that level, i.e.

$$[\Phi_n^a, \Phi_0^b] = \sum_{c=1}^{M} \sum_{l=0}^{n} D_{nl}^{abc} \Phi_l^c. \tag{7}$$

The whole constraint structure of the system is composed of the *constraint chains* via the relation

$$\Phi_n^a = [\Phi_{n-1}^a, H]. \tag{8}$$

Each chain begins with a primary constraint and is also labelled according to it, for example, by chain 'a' we mean the chain that begins with Φ_0^a. We assume that at each level the constraint chains are knitted simultaneously. In other words, after constructing all nth-level constraints Φ_n^a from the relation (8), one constructs the $(n+1)$th-level constraints Φ_{n+1}^a in the same way, and so on. However, during this procedure it is also possible that some chains terminate before others.

Suppose that the chain which begins with Φ_0^a, terminates after A steps. This means that no new constraint would emerge from the consistency condition of Φ_A^a. This would be possible only if the Poisson bracket of Φ_A^a with the canonical Hamiltonian is a combination of the previous constraints, i.e.

$$[\Phi_A^a, H] = \sum_{a'=1}^{M} \sum_{n'=0}^{A} F_{n'}^{aa'} \Phi_{n'}^{a'}. \tag{9}$$

Here, the index n' cannot exceed A', the *length* of the chain a'. For this reason we assume that $F_{n'}^{aa'}$ vanishes for $n' > A'$. To make these considerations simpler, we release the upper limit of summation over n' in (9) to be, say A_1, the length of the longest chain, but alternatively impose the following condition on the coefficients $F_{n'}^{aa'}$:

$$F_{n'}^{aa'} = 0 \qquad \text{if} \quad n' > A' \quad \text{or} \quad n' > A. \tag{10}$$

Regarding different lengths of the chains, care is also needed in relation (7). If we denote by C the length of the chain c, then it is obvious that no constraint Φ_l^c with $l > C$ should exist. Therefore, we impose the condition

$$D_{nl}^{abc} = 0 \qquad \text{if} \quad l > C \tag{11}$$

without changing the upper limit of summation over l. The technique of introducing vanishing coefficients instead of changing the limits, as in (10) and (11), would be useful in the next section.

Relations (7)–(9) give the complete constraint structure of our first-class system. The coefficients D and F in (7) and (9) are, in general, functions of phase space coordinates, and as in [9] we call them Hamiltonian constraint structure coefficients (HCSCs).

An important point to add is that the Hamiltonian constraints introduced above are not necessarily *irreducible*. In other words, it may happen that the constraints Φ_n^a, defined in (8), are not independent functions of phase space coordinates. If this is the case, one can reduce the number of constraints by choosing instead a smaller number of functions such that their vanishing implies that $\Phi_n^a = 0$. This has been done in [3, 11]. Alternatively, here we keep all the constraints in the same form that they emerge from relation (8), even though they can be *reducible*.

2.3. Two methods

As mentioned before, there are two methods to construct a gauge generator satisfying the conditions (6). In the first method [2–5], one assumes that the generator of GT, depends on a set of infinitesimal arbitrary function of time, $\epsilon_a(t)$, and their time derivatives of several orders. From (1), it is clear that there exist M (the number of primary first-class constraints) gauge degrees of freedom. So the number of $\epsilon_a(t)$ in G should also be M. One can suggest G in the following way:

$$G(q,p,t) = \sum_{a=1}^{M}\sum_{n=0}^{A} G^a_{A-n}(q,p)\frac{\mathrm{d}^n}{\mathrm{d}t^n}\epsilon^a(t) \tag{12}$$

where the upper limit A, in general, may depend on the index a. Then one can impose the following conditions on the coefficients G_k^a, in order that $G(q,p,t)$ satisfy the conditions (6):

$$\begin{aligned} &\left[G_k^a, \text{constraint}\right] = \text{constraint} \\ &\left[G_k^a, \text{PFC}\right] = \text{PFC} \\ &G_0^a = \text{PFC} \\ &\left[G_k^a, H\right] + G_{k+1}^a = \text{PFC} \\ &\left[G_A^a, H\right] = \text{PFC}. \end{aligned} \tag{13}$$

Now the problem is, given the set of primary and secondary constraints of a system, what is the set of G_k^a which satisfy the conditions (13). In other words, does the conditions (13) have any solution for G_k^a? In [3], a complicated procedure to obtain an answer is proposed. They show that *in principle* one can find a solution for (13) locally. However, some closed formulae that give G_k^a explicitly in terms of constraints of the system, seems inaccessible.

The second method [6], on the other hand, is based on the expansion of the generator $G(q,p,t)$ in terms of (first-class) constraints of the system, as follows:

$$G(q,p,t) = \sum_{a=1}^{M}\sum_{n=0}^{A} C_n^a(q,p,t)\Phi_n^a. \tag{14}$$

The coefficients $C_n^a(q,p,t)$ should be determined such that the conditions (6) hold for G.

According to its construction, G is apparently first class. The other two conditions in (6) impose the following conditions on C_n^a [6]:

$$\frac{\partial C_n^a}{\partial t} + \left[C_n^a, H\right] + \sum_{a'=1}^{M} C_{A'}^{a'} F_n^{a'a} + C_{n-1}^a = 0 \qquad \begin{cases} a = 1,\dots,M \\ n = 1,\dots,A \end{cases} \tag{15}$$

$$\sum_{a'=1}^{M}\sum_{n'=1}^{A'}\left(\left[C_{n'}^{a'}, \Phi_0^a\right] + \sum_{a''=1}^{M}\sum_{n''=n'}^{A'} C_{a''}^{n''} D_{n'n''}^{a'aa''}\right) = 0 \qquad a = 1,\dots,M. \tag{16}$$

For a definite chain a, there exist A equations from (15) for $(A+1)$ unknowns C_n^a. This enables one to determine all C_n^a $(n < A)$ in terms of, say, C_A^a. To this end, one should guess C_A^a in such a way that the resulting C_n^a satisfy the set of equations (16). However, this is not a simple task, in general. Therefore, this method has neither a closed-form answer for $G(q, p, t)$ in terms of the constraints, nor gives an explicit algorithm to obtain it in an arbitrary case. Moreover, the existence of an answer for C_n^a in (15) and (16) for a general case is an open question, as mentioned in [6]. However, in the same reference, it is shown that the generator $G(q, p, t)$ can be constructed using this method for systems having at most second-level constraints, and also for a system with the following conditions:

$$D_{nl}^{abc} = 0 \qquad l \neq 0 \tag{17}$$

$$F_{n'}^{aa'} = \text{constant}. \tag{18}$$

Although most of the physical examples of constraint systems such as Yang–Mills fall into the first category, nevertheless the system described in (17) and (18) provides a meaningful and non-trivial tool to construct the explicit form of the gauge generator $G(q, p, t)$ in terms of the constraints of a system and to compare the two methods mentioned above. That is what we will do in the following section.

3. Explicit form of the gauge generator

One can choose one of the methods mentioned in section 2 to derive the explicit form of the gauge transformation generator for the system described by relations (17) and (18). We choose the second method and try to find a solution for C_n^a.

As mentioned before, considering just equations (15), C_A^a are at our disposal. It is not difficult to see that if one chooses C_A^a as arbitrary functions of time, then considering (18), the other coefficients C_n^a would also be only functions of time (independent of phase space coordinates). This makes the Poisson brackets of coefficients C_n^a with H and primary constraints in (15) and (16) vanish. Therefore, equation (16), taking into account (17), is automatically satisfied. If one assumes

$$C_A^a = \eta^a(t) \tag{19}$$

then from (15), with $n = A - 1$, one obtains

$$C_{A-1}^a = -\frac{\mathrm{d}\eta^a}{\mathrm{d}t} - \sum_{a'=1}^{M} \eta^{a'} F_A^{a'a}. \tag{20}$$

Inserting (19) and (20) in (15) gives

$$C_{A-2}^a = (-1)^2 \frac{\mathrm{d}^2\eta^a}{\mathrm{d}t^2} + \sum_{a'=1}^{M} \left((-1)\eta^{a'} F_{A-1}^{a'a} + (-1)^2 \frac{\mathrm{d}\eta^{a'}}{\mathrm{d}t} F_A^{a'a} \right). \tag{21}$$

Repeating the same procedure, after l steps we have

$$C_{A-l}^a = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^l \eta^{a(t)} - \sum_{a'=1}^{M} \sum_{r=1}^{l} F_{A-l+r}^{a'a} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{r-1} \eta^{a'}(t) \qquad l = 1, \ldots, A. \tag{22}$$

One finally finds (after A steps) all the coefficients C_n^a. Inserting them into the expansion (15) for G gives

$$G(q, p, t) = \sum_{a=1}^{M} \left[\eta^a \Phi_A^a + \sum_{l=1}^{A} \left(\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^l \eta^a - \sum_{a'=1}^{M} \sum_{r=1}^{l} F_{A-l+r}^{a'a} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^r \eta^{a'} \right) \Phi_{A-l}^a \right]. \tag{23}$$

In this form, the expansion of the gauge generator is sorted in terms of the level of constraints. On the other hand, the coefficients are complicated combinations of M arbitrary functions of time, $\eta^a(t)$ and their derivatives. Part of the complexity is due to the variability of the upper limit of some summations in (23). However, this is not satisfactory, since sorting in terms of the order of time derivatives of gauge variables (or pure gauge fields in the field theory) is more acceptable than sorting in terms of the level of constraints. Constraints can essentially be redefined at each level [3], and one can construct several constraint structures without affecting the physical content of the system. Consequently, we prefer to change the generator (23) in the form of relation (12), to make it more meaningful.

To reach this goal, we investigate a simple technique to make the summations in (23) between fixed lower and upper limits. This trick enables us ultimately to change the order of summations and write $G(q, p, t)$ in the desired form.

Suppose the constraint chains are numbered in decreasing length order, i.e. the chain $a = 1$ is the longest. Let its length be A_1. One can make the length of all the chains equal by adding to them a set of *virtual* vanishing constraints. In this way all constraint chains have length A_1, and one can release the condition $a < A$, but alternatively for virtual constraints one should assume

$$\Phi_n^a \equiv 0 \qquad A < n \leqslant A_1. \tag{24}$$

We can now rewrite (15) with fixed upper limits as

$$G(q, p, t) = \sum_{a=1}^{M} \sum_{n=0}^{A_1} C_n^a \Phi_n^a \tag{25}$$

where for $n > A$ C_n^a can be chosen arbitrarily. Therefore, we assume that they satisfy the recursive equation (15) too. To solve the whole set of equations (15) for $n = 1, \ldots, A_1$, this time one should begin with coefficients $C_{A_1}^a$. Assuming

$$C_{A_1}^a = \epsilon^a(t) \tag{26}$$

and taking assumption (10) into account, equation (15) for the added coefficients, reads:

$$\frac{\partial C_n^a}{\partial t} + C_{n-1}^a = 0 \qquad n = A+1, \ldots, A_1 \tag{27}$$

whose solution is

$$C_{A_1-l}^a = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^l \epsilon^a(t) \qquad l = 0, \ldots, A_1 - A. \tag{28}$$

Note that the coefficients derived in (28) do not appear in the gauge generator (25), since they are multiplied by vanishing Φ_n^a (see equation (24)). Therefore, for a fixed label a, derivatives of $\epsilon^a(t)$ up to order $A_1 - A$ do not appear in G. In this manner, the function

$$C_A^a = \eta^a(t) = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{A_1-A} \epsilon^a(t) \tag{29}$$

is the first arbitrary function with label a that appears in the gauge generator. Considering (29), the remaining C_n^a are just derived in (22). It is only enough to introduce $\eta^a(t)$ from (29) in (22) to obtain

$$C_{A-l}^a = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{l+A_1-A} \epsilon^a(t) - \sum_{a'=1}^{M} \sum_{r=1}^{l} F_{A-l+r}^{a'a} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{A_1-A'+r-1} \epsilon^{a'}(t) l = 1, \ldots, A. \tag{30}$$

If we introduce the new variables

$$
\begin{aligned}
l' &= l + A_1 - A \\
J &= A_1 - A' + r - 1
\end{aligned}
\tag{31}
$$

and substitute l and r in terms of them; then the relation (30) reads

$$
\begin{aligned}
& C^a_{A_1-l'} = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{l'} \epsilon^a(t) - \sum_{a'=1}^{M} \sum_{J=A_1-A'}^{A-A'+l'-1} F^{a'a}_{A'-l'+J+1} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{J} \epsilon^{a'}(t) \\
& l' = A_1 - A + 1, \ldots, A_1.
\end{aligned}
\tag{32}
$$

To have non-vanishing $F^{a'a}_{\cdots}$ in the above relation, from (10), one condition is

$$
J \leqslant l' - 1 < A_1.
$$

Now regarding the upper limit of J in the corresponding summation in (32), two things may happen. If $A - A' + l' - 1 > A_1$, then one can truncate the summation at A_1. (Nevertheless, it is possible that the coefficients $F^{a'a}_{\cdots}$ in the summand begin to vanish before J touches the upper limit, which is not disturbing.) On the other hand, if $A - A' + l' - 1 < A_1$, one can increase the upper limit of summation over J to A_1. That is because, for $J = A - A' + l' - 1$, the second term in (32) would contain the coefficient $F^{a'a}_{A}$, and exceeding J would give vanishing $F^{a'a}_{n}$ which has no effect on the relation. Consequently, we are allowed to change (32) into the following form:

$$
\begin{aligned}
& C^a_{A_1-l'} = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{l'} \epsilon^a(t) - \sum_{a'=1}^{M} \sum_{J=A_1-A'}^{A_1} F^{a'a}_{A'-l'+J+1} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{J} \epsilon^{a'}(t) \\
& l' = A_1 - A + 1, \ldots, A_1.
\end{aligned}
\tag{33}
$$

Now, considering (24), the gauge generator (25) can be written as

$$
G(q, p, t) = \sum_{a=1}^{M} \sum_{l'=A_1-A}^{A_1} C^a_{A_1-l'} \Phi^A_{A_1-l'}. \tag{34}
$$

Inserting (33) into (34) gives

$$
\begin{aligned}
G(q, p, t) = {} & \sum_{a=1}^{M} \sum_{l'=A_1-A}^{A_1} \Phi^a_{A_1-l'} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{l'} \epsilon^a \\
& - \sum_{a=1}^{M} \sum_{l'=A_1-A}^{A_1} \sum_{a'=1}^{M} \sum_{J=A_1-A'}^{A_1} F^{a'a}_{A'-l'+J+1} \Phi^a_{A_1-l'} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{J} \epsilon^{a'}.
\end{aligned}
\tag{35}
$$

In the second term above, we have two similar summations over a and a' (followed by subsequent similar summations over l' and J), which can be easily interchanged. In fact, this is the main reason for trying to derive the relation (33) as it is. Changing the order of the summations, and then exchanging the names $a' \leftrightarrow a$ and $l' \leftrightarrow J$ (just in the second term of (35)) gives

$$
\begin{aligned}
G(q, p, t) = {} & \sum_{a=1}^{M} \sum_{l'=A_1-A}^{A_1} \Phi^a_{A_1-l'} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{l'} \epsilon^a \\
& - \sum_{a=1}^{M} \sum_{l'=A_1-A}^{A_1} \sum_{a'=1}^{M} \sum_{J=A_1-A'}^{A'} F^{aa'}_{A-J+l'+1} \Phi^{a'}_{A_1-J} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{l'} \epsilon^{a}.
\end{aligned}
\tag{36}
$$

This can be rewritten in the form we desired, i.e.

$$G(q, p, t) = \sum_{a=1}^{M} \sum_{l'=A_1-A}^{A_1} G^a_{A_1-l'} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{l'} \epsilon^a(t) \tag{37}$$

where

$$G^a_{A_1-l'} = \Phi^a_{A_1-l'} - \sum_{a'=1}^{M} \sum_{J=A_1-A'}^{A_1} F^{a'a}_{A-J+l'+1} \Phi^{a'}_{A_1-J}. \tag{38}$$

Finally, to simplify the form of gauge generator (37) by lowering the order of time derivatives, it is better to replace $\epsilon^a(t)$ with the set of arbitrary functions $\eta^a(t)$ introduced in (29) and also again use the index l instead of l' (see equation (31)). The result is

$$G(q, p, t) = \sum_{a=1}^{M} \sum_{l=0}^{A} G^a_{A-l} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{l} \eta^a(t) \tag{39}$$

where

$$G^a_k = \Phi^a_k - \sum_{a'=1}^{M} \sum_{s=0}^{A'} F^{a'a}_{(A-k)+s+1} \Phi^{a'}_s. \tag{40}$$

Relations (39) and (40) are our final result for the gauge transformation generator. In this form one can easily observe the way arbitrary functions of time and their derivatives appear in a gauge transformation. As can be seen from (40) the coefficients G^a_k are some definite combinations of (first-class) constraints of the system.

The final task is to be sure that the set of coefficient G^a_k in (39), viewed as the coefficients of expansion of the gauge generator in the first method (relation (12)), do really satisfy the conditions (13). Since G^a_k are totally composed of (first-class) constraints, they are first-class objects and the first condition of (13) is satisfied. Moreover, from assumption (17) it is obvious that their Poisson brackets with PFCs are PFC. So the second condition of (13) is also satisfied. To verify the third condition, one can write

$$G^a_0 = \Phi^a_0 - \sum_{a'=1}^{M} \sum_{s=0}^{A'} F^{a'a}_{A+s+1} \Phi^{a'}_s.$$

The second term above vanishes due to (10), showing that $G^a_0 = \Phi^a_0 = \mathrm{PFC}$.

The fourth condition can be verified by a simple calculation. Using (8) it is not difficult to see that

$$\left[G^a_k, H\right] - G^a_{k+1} = \sum_{a'=1}^{M} F^{aa'}_{A-k} \Phi^{a'}_0 = \mathrm{PFC}.$$

Finally, to verify the last condition of (13), using (10) one can write

$$G^a_A = \Phi^a_A - \sum_{a'=1}^{M} \sum_{s=0}^{A'-1} F^{aa'}_{s+1} \Phi^{a'}_s.$$

Then, direct calculation by use of (8) and (9) shows that

$$\left[G^a_A, H\right] = \sum_{a'=1}^{M} F^{aa'}_0 \Phi^{a'}_0 = \mathrm{PFC}.$$

In this way, we see that the gauge generator (39) with coefficients G^a_k given in (40), is indeed a solution of conditions (13) of the first method. Consequently, beginning with equations (15) and (16) for the coefficients C^a_n of the gauge generator in the second method, we have succeeded in finding a gauge generator with the first method.

4. Example

Consider the Lagrangian

$$L = \dot{x}\dot{z} + z\dot{x} + yz + xz. \tag{41}$$

The momenta are

$$\begin{aligned} p_x &= \dot{z} + z \\ p_y &= 0 \\ p_z &= \dot{x}. \end{aligned} \tag{42}$$

The primary constraint of the system is

$$\Phi_0 = p_y. \tag{43}$$

The total Hamiltonian reads

$$H_T = P_x P_z - z P_z - yz - zx + \lambda p_y. \tag{44}$$

The secondary constraints are derived from (8) as follows:

$$\begin{aligned} \Phi_1 &= [p_y, H_T] = z \\ \Phi_2 &= [z, H_T] = p_x - z. \end{aligned} \tag{45}$$

There is only one constraint chain, and the chain indices $a, a', \ldots$ can be dropped. The constraint Φ_2 is the terminating element of the chain (i.e. $A = 2$). Its Poisson bracket with Hamiltonian reads

$$[\Phi_2, H] = 2z - p_x = \Phi_1 - \Phi_2. \tag{46}$$

Comparing this relation with (9) shows that $F_0 = 0$, $F_1 = 1$ and $F_2 = -1$.

On the other hand, the Poisson brackets of constraints with each other vanishes identically. Consequently, from (7) we have $D_{nl} = 0$ for all n and l. Now everything is prepared to write down the gauge generator, which can be written from (39) in the following form:

$$G(q, p, t) = G_0 \frac{\mathrm{d}^2\eta}{\mathrm{d}t^2} - G_1 \frac{\mathrm{d}\eta}{\mathrm{d}t} + G_2 \eta. \tag{47}$$

From (40) and the data regarding Φ_k and F_n, one reads

$$\begin{aligned} G_0 &= \Phi_0 = p_y \\ G_1 &= \Phi_1 - F_2\Phi_0 = z + p_y \\ G_2 &= \Phi_2 - F_1\Phi_0 - F_2\Phi_1 = p_x - p_y. \end{aligned} \tag{48}$$

Therefore, the gauge transformation generator is

$$G(q, p, t) = \ddot{\eta} p_y - \dot{\eta}(p_y + z) + \eta(p_x - p_y). \tag{49}$$

It is not difficult to find the gauge variations of the coordinates and momenta from (3) and (49) and then verify that they really transform a solution of equations of motion into another solution.

To see, in this simple example, what has happened behind the lengthy derivation of the coefficients G_k^a in (39), one can begin again with relation (14) for the gauge generator:

$$G(q, p, t,) = \sum_{n=0}^{2} C_n \Phi_n. \tag{50}$$

Then, equations (15) for C_n read

$$\begin{aligned} C_1 &= -\frac{\partial C_2}{\partial t} - [C_2, H] + C_2 \\ C_0 &= -\frac{\partial C_1}{\partial t} - [C_1, H] - C_2. \end{aligned} \tag{51}$$

Assuming $C_2 = \eta(t)$ one obtains

$$\begin{aligned} C_1 &= -\dot{\eta} + \eta \\ C_0 &= \ddot{\eta} - \dot{\eta} - \eta. \end{aligned} \tag{52}$$

Inserting (52) into (50) and sorting the result in the order of time derivatives (which is the main point), will again give the gauge generator (49).

5. Concluding remarks

In this paper we proceed to the problem of the relationship between gauge transformations and first-class constraints of a system. We first reviewed the conditions that the generator of gauge transformations should obey (conditions (6)). Considering the gauge generator in the form $G = \sum G_n^a (\mathrm{d}^n \epsilon^a / \mathrm{d}t^n)$, some authors [4, 5, 10] have equated G_n^a with constraints of different levels (primary, secondary, etc) and then derived the gauge generator with the required conditions. This method, even though it works for some simple constrained systems such as Yang–Mills, is not suitable for the general case.

This difficulty has been observed in the literature. In [3] the authors proved that in principle there exists a procedure to find the coefficients G_n^a in terms of constraints of the system. A closed and clear result is, however, far from being reached with this method. In [6] a gauge generator of the form $G = \sum C_n^a(q, p, t)\, \Phi_n^a(q, p)$, where Φ_n^as are constraints of different levels, is considered. This generator is written directly in terms of the constraints, but unfortunately the conditions on C_n^a (which originates from the conditions on G), are difficult to handle. Furthermore, in the general case it seems impossible to prove the existence of any solution for the C_n^a.

However, in our opinion the above-mentioned difficulties should not prevent one from getting a general idea of how the generator of gauge transformations depends on the constraint structure of a system. For this reason, we considered a special class of constrained systems (described in relations (17) and (18) above). Then writing the gauge generator in the form $G = \sum C_n^a \Phi_n^a$, we constructed a consistent solution for C_n^a in terms of a different order of time derivatives of arbitrary functions $\epsilon^a(t)$. Then, changing the order of summation (with some technical details), we found the gauge generator in the form $G = \sum G_n^a (\mathrm{d}^n \epsilon^a / \mathrm{d}t^n)$, where G_n^a are expressed explicitly in terms of constraints of different levels (relations (39) and (40)).

We think that this work can better illuminate the relationship between gauge transformations and first-class constraints of the system. We emphasize (and our investigation in this paper shows) that finding a closed result for the gauge generator in terms of constraints of a system seems impossible for the generic case. However, it seems better to pay the price of considering some limitations on the constraint algebra, in order to observe explicitly the interesting relationship between gauge symmetry and constraint structure.

As far as we know, except for some simple systems with just one or two levels of constraints, our results is the clearest and most explicit formula for the gauge generator. One way to extend the present result toward the most general case would be to find an algorithm to redefine the constraints of an arbitrary system in such a way that they satisfy the conditions (17) and (18), which we considered here.

Acknowledgment

The authors like to thank M Loran for reading the manuscripts.

References

[1] Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Yeshiva University Press)
[2] Gracia X and Pons J M 1988 *Ann. Phys.* **187** 355
[3] Gomis J, Henneaux M and Pons J M 1990 *Class. Quantum Grav.* **7** 1089
[4] Castellani L 1982 *Ann. Phys., NY* **143** 357
[5] Anderson J L and Bergman P G 1951 *Phys. Rev.* **83** 1018
[6] Cabo A, Chaichian M and Louis Martinez D 1993 *J. Math. Phys.* **34** 5646
[7] Shirzad A 1998 *J. Phys. A: Math. Gen.* **31** 2747
[8] Chaichian M and Louis Martinez D 1994 *J. Math. Phys.* **35** 6536
[9] Shirzad A and Saadeghnezhaad N 1998 *J. Phys. A: Math. Gen.* **31** 7403
[10] Costa M E V, Girotti H O and Simoes T J M 1985 *Phys. Rev.* D **32** 405
[11] Henneaux M, Teitelboim C and Zanelli J 1990 *Nucl. Phys.* B **332** 169